

Valadier-like formulas for the supremum function I^*

R. Correa[†], A. Hantoute[‡], M.A. López[§]

[†]Universidad de O'Higgins, Chile, and DIM-CMM of Universidad de Chile

[‡]Center for Mathematical Modeling (CMM), Universidad de Chile

[§]Universidad de Alicante, Spain, and CIAO, Federation University, Ballarat, Australia

Abstract

We generalize and improve the original characterization given by Valadier [18, Theorem 1] of the subdifferential of the pointwise supremum of convex functions, involving the subdifferentials of the data functions at nearby points. We remove the continuity assumption made in that work and obtain a general formula for such a subdifferential. In particular, when the supremum is continuous at some point of its domain, but not necessarily at the reference point, we get a simpler version which gives rise to the Valadier formula. Our starting result is the characterization given in [11, Theorem 4], which uses the ε -subdifferential at the reference point.

Key words. Pointwise supremum function, convex functions, Fenchel subdifferential, Valadier-like formulas.

Mathematics Subject Classification (2010): 26B05, 26J25, 49H05.

1 Introduction

Consider a family of extended real-valued convex functions $f_t : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, indexed by an arbitrary index set T , and defined in a locally convex topological vector space X . Under the continuity of the supremum function

$$f := \sup_{t \in T} f_t \tag{1}$$

at a point $x \in X$, a remarkable pioneering result due to Valadier [18] states that the subdifferential of f at x is completely characterized by means of the subdifferentials of

*Research of the first and the second authors is supported by CONICYT grants, Fondecyt no. 1150909 and 1151003, Basal PFB-03 and Basal FB003. Research of the first and third authors supported by MINECO of Spain and FEDER of EU, grant MTM2014-59179-C2-1-P. Research of the third author is also partially supported by the Australian Research Council: Project DP160100854.

[†]e-mail: rcorrea@dim.uchile.cl

[‡]e-mail: ahantoute@dim.uchile.cl (corresponding author)

[§]e-mail: marco.antonio@ua.es

data functions f_t at nearby points; more precisely

$$\partial f(x) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x), p(y-x) \leq \varepsilon} \partial f_t(y) \right\}, \quad (2)$$

where $\mathcal{P}$ is the family of continuous seminorms in X , and

$$T_\varepsilon(x) := \{t \in T \mid f_t(x) \geq f(x) - \varepsilon\}; \quad (3)$$

when $f(x) = +\infty$, we have $f_t(x) = +\infty$ for all $t \in T_\varepsilon(x)$. We also use the notation

$$T(x) := \{t \in T \mid f_t(x) = f(x)\}. \quad (4)$$

There have been many contributions to this problem, in various settings, depending on the structure of the space X , the algebraic and topological properties of the index set T , the behavior of the function $f(x, t) := f_t(x)$, etc. See, for instance, [2, 3, 4, 10, 6, ?, 14, 17, 19], among many others.

More recently, using the machinery of ε -subdifferentials, and under the following condition

$$\text{cl } f = \sup_{t \in T} \text{cl } f_t, \quad (5)$$

involving the lower semicontinuous (lsc, for short) envelopes, it has been established in [11, Theorem 4] (see also [9]) that, for every $x \in X$,

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \{N_{L \cap \text{dom } f}(x)\} \right\}, \quad (6)$$

where

$$\mathcal{F}(x) := \{L \subset X \mid L \text{ is a finite-dimensional linear subspace such that } x \in L\}. \quad (7)$$

When T is finite and $T(x) = T$, the last formula gives rise to ([2]; see, also, [11, Corollary 12])

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T(x)} \partial_\varepsilon f_t(x) \right\}, \quad (8)$$

being the first result of this kind that does not require any continuity conditions.

Both approaches are based on enlargements of ∂f_t , using either the exact subdifferentials at nearby points or the ε -subdifferentials at the reference point. They are comparable because: (i) both formulas coincide when the f_t 's are affine, (ii) both enlargements are related in the Banach spaces setting thanks to the Brøndsted-Rockafellar theorem.

The main purpose of the present paper consists of generalizing and improving formula

(2). We drop the continuity assumption and obtain general formulas for $\partial f(x)$, in the settings of Banach spaces (16) and locally convex spaces (24), both of them involving enlargements of ∂f_t , which use the exact subdifferentials of the f_t 's at nearby points.

The paper is organized as follows. After Section 2, devoted to notation and preliminaries, Section 3 provides the free-continuity extension of (2) in a Banach space (16), via the application of Brøndsted-Rockafellar theorem. Our characterization is comparable with the main limiting-like results given in [15]. The validity limitation of this result outside the Banach setting is illustrated in this section by means of some examples. In Section 4 we establish the second free-continuity formula for ∂f (24) in the setting of locally convex spaces. Section 5 is devoted to deriving formula (38), which constitutes a simpler version of (16) and (24), which is valid only when we are confined to the case in which the supremum function f is finite and continuous at some point. This last section finishes with the derivation of (2) from our formula (38).

In a forthcoming paper we establish the corresponding counterparts of the formulas provided in the current paper to the case in which the index set T is compact and the functions $t \rightarrow f_t(z)$, $z \in \text{dom } f$, are upper semicontinuous on $T_\varepsilon(x)$. This will generalize the second theorem in [18]. In the same paper, some applications to the characterization of normal cones to sublevel sets are also given, extending some recent results in [12].

2 Notation and preliminaries

In this paper X stands for a (real) separated locally convex (lcs, shortly) space, whose topological dual space is denoted by X^* and, unless otherwise stated, is endowed with the weak*-topology. Hence, the (X, X^*) forms a dual topological pair by means of the canonical bilinear form $\langle x, x^* \rangle = \langle x^*, x \rangle := x^*(x)$, $(x, x^*) \in X \times X^*$. The zero vectors in all the involved spaces are denoted by θ , and the convex, closed and balanced neighborhoods of θ are called θ -neighborhoods. The families of such θ -neighborhoods in X and in X^* are denoted by $\mathcal{N}_X$ and $\mathcal{N}_{X^*}$, respectively, whereas for $x \in X$ and $x^* \in X^*$ we denote $\mathcal{N}_x := x + \mathcal{N}_X$ and $\mathcal{N}_{x^*} := x^* + \mathcal{N}_{X^*}$.

Given a nonempty set A in X (or in X^*), by $\text{co } A$, $\text{cone } A$, and $\text{aff } A$ we denote the *convex hull*, the *conic hull*, and the *affine hull* of A , respectively. Moreover, $\text{int } A$ is the *interior* of A , and $\text{cl } A$ and $\overline{A}$ are indistinctly used for denoting the *closure* of A (weak*-closure if $A \subset X^*$). Thus, $\overline{\text{co}} A := \text{cl}(\text{co } A)$, $\overline{\text{aff}} A := \text{cl}(\text{aff } A)$, etc. We use $\text{ri } A$ to denote the (topological) *relative interior* of A (i.e., the interior of A in the topology relative to $\overline{\text{aff}} A$). Associated with a nonempty subset A of X , we consider the (one-sided) *polar* and the *orthogonal* of A defined, respectively, by

$$A^\circ := \{x^* \in X^* \mid \langle x^*, x \rangle \leq 1 \text{ for all } x \in A\},$$

and

$$A^\perp := \{x^* \in X^* \mid \langle x^*, x \rangle = 0 \text{ for all } x \in A\}.$$

For $B \subset X$ and $\Omega \subset \mathbb{R}$ we define

$$\begin{aligned} A + B &:= \{a + b \mid a \in A, b \in B\}, \\ \Omega A &:= \{\lambda a \mid \lambda \in \Omega, a \in A\}, \end{aligned} \tag{9}$$

with the conventions

$$\Omega \emptyset = A + \emptyset = \emptyset. \tag{10}$$

Throughout the paper we apply the following two lemmas concerning product (9). Let us introduce the set

$$\Delta_k := \{(\lambda_1, \dots, \lambda_k) \mid \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1\}.$$

Lemma 1 *If Ω is a compact interval such that $0 \notin \Omega$, then*

$$\Omega(\overline{\text{co}}A) = \overline{\text{co}}(\Omega A).$$

Proof. The conclusion follows from the algebraic equality $\Omega(\text{co } A) = \text{co}(\Omega A)$ together with its consequence

$$\Omega(\overline{\text{co}}A) = \text{cl}(\Omega(\text{co } A)) = \overline{\text{co}}(\Omega A),$$

both being true thanks to the condition $0 \notin \Omega$. ■

Lemma 2 *Suppose that $(\Lambda_\varepsilon)_{\varepsilon>0}$ is a non-increasing family of closed sets in $\mathbb{R}$ (i.e. $\varepsilon' < \varepsilon$ implies $\Lambda_{\varepsilon'} \subset \Lambda_\varepsilon$) such that $\bigcap_{\varepsilon>0} \Lambda_\varepsilon = \{1\}$. Let $(A_\varepsilon)_{\varepsilon>0}$ be another non-increasing family of closed sets in X (or in X^*). Then*

$$\bigcap_{\varepsilon>0} \Lambda_\varepsilon A_\varepsilon = \bigcap_{\varepsilon>0} A_\varepsilon.$$

Proof. The inclusion " $\supset$ " is obvious as $1 \in \Lambda_\varepsilon$ entails $A_\varepsilon \subset \Lambda_\varepsilon A_\varepsilon$. Let us prove the inclusion " $\subset$ ".

If $a \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon A_\varepsilon$, we have $a = \lambda_\varepsilon a_\varepsilon$ with $\lambda_\varepsilon \in \Lambda_\varepsilon$ and $a_\varepsilon \in A_\varepsilon$, for all $\varepsilon > 0$. Since $\lim_{\varepsilon \downarrow 0} \lambda_\varepsilon = 1$ we can assume that $\lambda_\varepsilon > 0$. For each $\varepsilon > 0$ the net $(a_\delta)_{0 < \delta < \varepsilon}$ is contained in A_ε , and due to the closedness of A_ε , $\lim_{\delta \downarrow 0} a_\delta = \lim_{\delta \downarrow 0} (\lambda_\delta)^{-1} a = a \in A_\varepsilon$, and we are done. ■

We say that a function $\varphi : X \longrightarrow \overline{\mathbb{R}}$ is proper if its (effective) domain, $\text{dom } \varphi := \{x \in X \mid \varphi(x) < +\infty\}$, is nonempty. We say that φ is *convex* (lower semicontinuous or *lsc*, for short, respectively) if its *epigraph*, $\text{epi } \varphi := \{(x, \lambda) \in X \times \mathbb{R} \mid \varphi(x) \leq \lambda\}$, is convex (closed, respectively). We shall denote by $\Lambda(X)$ the set of all the proper convex functions defined on X and by $\Gamma_0(X)$ the functions in $\Lambda(X)$ which are *lsc*. The *lsc envelope* of φ is the function $\text{cl } \varphi$ such that $\text{epi}(\text{cl } \varphi) = \text{cl}(\text{epi } \varphi)$.

If $\varepsilon \geq 0$, the ε -subdifferential of φ at a point x where $\varphi(x)$ is finite is the weak*-closed convex set

$$\partial_\varepsilon \varphi(x) := \{x^* \in X^* \mid \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle - \varepsilon \text{ for all } y \in X\}.$$

If $\varphi(x) \notin \mathbb{R}$, then we set $\partial_\varepsilon \varphi(x) := \emptyset$. In particular, for $\varepsilon = 0$ we get the *Fenchel subdifferential* of φ at x , $\partial \varphi(x) := \partial_0 \varphi(x)$. When $\partial \varphi(x) \neq \emptyset$, we know that

$$\varphi(x) = (\text{cl } \varphi)(x) \text{ and } \partial \varphi(x) = \partial(\text{cl } \varphi)(x). \quad (11)$$

The *support function*, the *indicator function* and the *Minkowski gauge* of $A \subset X$ are, respectively, defined as

$$\sigma_A(x^*) := \sup\{\langle x^*, a \rangle \mid a \in A\}, \text{ for } x^* \in X^*, \text{ and } \sigma_\emptyset \equiv -\infty, \quad (12)$$

$$I_A(x) := 0, \text{ if } x \in A; \quad +\infty, \text{ if } x \in X \setminus A,$$

and

$$p_A(x) := \inf\{\lambda \geq 0 : x \in \lambda A\},$$

with $\inf \emptyset = +\infty$. If $U \in \mathcal{N}_X$ (or $U \in \mathcal{N}_{X^*}$), then p_U is a continuous seminorm and

$$U = \{x \in X \mid p_U(x) \leq 1\}. \quad (13)$$

(Remember that U is closed.)

If A is convex, $x \in X$ and $\varepsilon \geq 0$, we define the ε -normal set to A at x as

$$N_A^\varepsilon(x) := \{x^* \in X^* \mid \langle x^*, y - x \rangle \leq \varepsilon \text{ for all } y \in A\}, \text{ if } x \in A; \quad \emptyset, \text{ otherwise.}$$

If $\varepsilon = 0$, we omit the reference to ε and write $N_A(x)$.

We shall frequently use the following well-known relation, for any set $A \subset X$

$$\bigcap_{U \in \mathcal{N}_X} \text{cl}(A + U) = \text{cl}(A). \quad (14)$$

The same relation is true for $A \subset X^*$ replacing $\mathcal{N}_X$ by $\mathcal{N}_{X^*}$.

Lemma 3 *Let A be a set in X^* . Then for every $x \in X$*

$$\bigcap_{L \in \mathcal{F}(x)} \text{cl}(A + L^\perp) = \text{cl}(A).$$

If X is a normed space, then

$$\bigcap_{\varepsilon > 0} \text{cl}(A + \varepsilon B_*) = \text{cl}(A),$$

where B_ is the closed unit ball in X^* .*

Proof. Due to the fact that the family of sets

$$\{x^* \in X^* \mid |\langle x_i, x^* \rangle| \leq \delta, \ i \in \overline{1, k}\}, \quad x_i \in X, \ i \in \overline{1, k}, \ \delta > 0\}$$

is a basis of θ -neighborhoods of the weak*-topology, and observing that

$$\{x_i \in X, i \in \overline{1, k}; x\}^\perp \subset \{x^* \in X^* \mid |\langle x_i, x^* \rangle| \leq \delta, i \in \overline{1, k}\},$$

(14) allows us to write

$$\text{cl}(A) \subset \bigcap_{L \in \mathcal{F}(x)} \text{cl}(A + L^\perp) \subset \bigcap_{U \in \mathcal{N}} \text{cl}(A + U) = \text{cl}(A).$$

Now, assume that X is a normed space. Since the topology on the norm in X^* is stronger than every compatible topology in X^* , we get, also using (14) in X^* ,

$$\text{cl}(A) \subset \bigcap_{\varepsilon > 0} \text{cl}(A + \varepsilon B_*) \subset \bigcap_{U \in \mathcal{N}_{X^*}} \text{cl}(A + U) = \text{cl}(A).$$

■

3 The case of Banach spaces

Our purpose in this section is to provide a first direct consequence of formula (6), which yields a free-continuity extension of (2) in a Banach space $(X, \|\cdot\|)$. This is accomplished via a straightforward application of the Brøndsted-Rockafellar theorem. We also illustrate by examples the validity limitation of this approach outside the Banach setting. We need the following previous simple lemma.

Lemma 4 *Given a convex function $\varphi : X \rightarrow \overline{\mathbb{R}}$, for every $x \in \text{dom } \varphi$ we have*

$$\bigcap_{L \in \mathcal{F}(x)} \partial(\varphi + \text{I}_L)(x) = \partial\varphi(x).$$

For a function $\varphi \in \Gamma_0(X)$, we recall here the mapping $\check{\partial}^\varepsilon \varphi : X \rightrightarrows X^*$ defined for $x \in \text{dom } \varphi$ as (see, e.g., [8, (7) in p. 1268])

$$\check{\partial}^\varepsilon \varphi(x) := \left\{ y^* \in X^* \mid \begin{array}{l} \exists y \in B_\varepsilon(x), \text{ such that } |\varphi(y) - \varphi(x)| \leq \varepsilon, \\ y^* \in \partial\varphi(y) \text{ and } |\langle y^*, y - x \rangle| \leq \varepsilon \end{array} \right\}, \quad (15)$$

which constitutes an enlargement of the Fenchel subdifferential, and that can also be written as

$$\check{\partial}^\varepsilon \varphi(x) = \bigcup_{y \in B_\varphi(x, \varepsilon)} \partial\varphi(y) \cap N_{\{y\}}^\varepsilon(x),$$

where

$$B_\varphi(x, \varepsilon) := \{y \in B_\varepsilon(x) \mid |\varphi(y) - \varphi(x)| \leq \varepsilon\}.$$

When $x \notin \text{dom } \varphi$ we adopt the natural convention $\check{\partial}^\varepsilon \varphi(x) := \emptyset$. It is clear that $\partial\varphi(x) \subset \check{\partial}^\varepsilon \varphi(x)$ and, moreover, it can be easily checked that $\bigcap_{\varepsilon > 0} \check{\partial}^\varepsilon \varphi(x) = \partial\varphi(x)$. Let us empha-

size here that the elements of $\check{\partial}^\varepsilon \varphi(x)$ are exact Fenchel subgradients at nearby points to x , not ε -subgradients like in (6) and (8).

The following version of the Brøndsted-Rockafellar theorem (see, e.g. [19, Theorem 3.1.1]) is a key tool in our approach.

Lemma 5 *Let $\varphi \in \Gamma_0(X)$ and $x \in \text{dom } \varphi$. Then, for every $\varepsilon > 0$ and $x^* \in \partial_\varepsilon \varphi(x)$, there exist $x_\varepsilon \in X$, $y_\varepsilon^* \in B_*$, and $\lambda_\varepsilon \in [-1, 1]$ such that*

$$\|x_\varepsilon - x\| \leq \sqrt{\varepsilon},$$

$$x_\varepsilon^* := x^* + \sqrt{\varepsilon}(y_\varepsilon^* + \lambda_\varepsilon x^*) \in \partial \varphi(x_\varepsilon),$$

$$|\langle x_\varepsilon^*, x_\varepsilon - x \rangle| \leq \varepsilon + \sqrt{\varepsilon}, \quad |\varphi(x_\varepsilon) - \varphi(x)| \leq \varepsilon + \sqrt{\varepsilon},$$

which implies that $x_\varepsilon^* \in \partial_{2\varepsilon} \varphi(x)$ and $x_\varepsilon^* \in \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} \varphi(x)$.

Now we give the main result in this section.

Theorem 6 *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t \in T} f_t$. Then for every $x \in X$*

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\}, \quad (16)$$

where $T_\varepsilon(x)$ and $\mathcal{F}(x)$ have been defined in (3) and (7), respectively.

Proof. If $x \notin \text{dom } f$ the formula holds trivially since both sets $N_{L \cap \text{dom } f}(x)$ and $\partial f(x)$ are empty and we apply (10); hence, we suppose that $x \in \text{dom } f$.

Step 1. To prove the inclusion “ $\supset$ ” we first show that, for $\varepsilon > 0$, we have

$$\bigcup_{t \in T_\varepsilon(x)} \check{\partial}^\varepsilon f_t(x) \subset \partial_{3\varepsilon} f(x). \quad (17)$$

Indeed, fix $t \in T_\varepsilon(x)$ and pick a $y^* \in \check{\partial}^\varepsilon f_t(x)$. By (15), there exists $y \in B_\varepsilon(x)$ with $|f_t(y) - f_t(x)| \leq \varepsilon$ such that $y^* \in \partial f_t(y)$ and $|\langle y^*, y - x \rangle| \leq \varepsilon$. Then for every $z \in X$

$$\begin{aligned} f(z) - f(x) &\geq f_t(z) - f_t(x) - \varepsilon \geq f_t(z) - f_t(y) - 2\varepsilon \\ &\geq \langle y^*, z - y \rangle - 2\varepsilon = \langle y^*, z - x \rangle + \langle y^*, x - y \rangle - 2\varepsilon \\ &\geq \langle y^*, z - x \rangle - 3\varepsilon, \end{aligned}$$

and we get $y^* \in \partial_{3\varepsilon} f(x)$.

Step 2. Now (15) implies, for every $L \in \mathcal{F}(x)$,

$$\begin{aligned} \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) &\subset \partial_{3\varepsilon} f(x) + N_{L \cap \text{dom } f}(x) \\ &\subset \partial_{3\varepsilon}(f + I_{L \cap \text{dom } f})(x) = \partial_{3\varepsilon}(f + I_L)(x), \end{aligned}$$

so that, by convexifying and intersecting over ε and L , and applying Lemma 4,

$$\begin{aligned} \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\} &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \partial_{3\varepsilon}(f + I_L)(x) \\ &= \bigcap_{L \in \mathcal{F}(x)} \partial(f + I_L)(x) \\ &= \partial f(x). \end{aligned}$$

Step 3. To prove the inclusion “ $\subset$ ” we pick an $x^* \in \partial_\varepsilon f_t(x)$ for $\varepsilon > 0$ and $t \in T_\varepsilon(x)$. Then by Lemma 5 there are $x_\varepsilon \in X$, $y_\varepsilon^* \in B_*$, and $\lambda_\varepsilon \in [-1, 1]$ such that

$$\|x_\varepsilon - x\| \leq \sqrt{\varepsilon},$$

$$x_\varepsilon^* := x^* + \sqrt{\varepsilon}(y_\varepsilon^* + \lambda_\varepsilon x^*) \in \partial f_t(x_\varepsilon),$$

$$|\langle x_\varepsilon^*, x_\varepsilon - x \rangle| \leq \varepsilon + \sqrt{\varepsilon}, \quad |f_t(x_\varepsilon) - f_t(x)| \leq \varepsilon + \sqrt{\varepsilon},$$

and

$$x_\varepsilon^* \in \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x).$$

Assuming without loss of generality that $1 \leq 2(1 - \sqrt{\varepsilon})$ (i.e., $\varepsilon < 1/4$), we have

$$x^* \in \frac{1}{1 + \lambda_\varepsilon \sqrt{\varepsilon}} \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + \frac{\sqrt{\varepsilon}}{1 + \lambda_\varepsilon \sqrt{\varepsilon}} B_* \subset \Lambda_\varepsilon \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + 2\sqrt{\varepsilon} B_*,$$

where $\Lambda_\varepsilon := \left[\frac{1}{1 + \sqrt{\varepsilon}}, \frac{1}{1 - \sqrt{\varepsilon}} \right]$. Hence, taking into account Lemma 3, formula (6) leads us to

$$\begin{aligned} \partial f(x) &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \Lambda_\varepsilon \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + 2\sqrt{\varepsilon} B_* + N_{L \cap \text{dom } f}(x) \right\} \\ &\subset \bigcap_{\delta > 0} \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \Lambda_\varepsilon \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + \delta B_* + N_{L \cap \text{dom } f}(x) \right\} \\ &= \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \bigcap_{\delta > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \Lambda_\varepsilon \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + \delta B_* + N_{L \cap \text{dom } f}(x) \right\} \\ &= \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \Lambda_\varepsilon \check{\partial}^{\varepsilon + \sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right\}. \end{aligned} \tag{18}$$

Step 4. Taking into account now that $N_{L \cap \text{dom } f}(x)$ is a cone we have

$$\Lambda_\varepsilon \check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) = \Lambda_\varepsilon \left(\check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right),$$

and (18) yields

$$\begin{aligned} \partial f(x) &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \Lambda_\varepsilon \left(\check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right) \right\} \\ &= \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \Lambda_\varepsilon \left(\bigcup_{t \in T_\varepsilon(x)} \check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right) \right\} \\ &= \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \Lambda_\varepsilon \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right\} \\ &= \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right\} \end{aligned} \quad (19)$$

where the second and the third equalities come from Lemma 1 and Lemma 2, respectively. Hence,

$$\begin{aligned} \partial f(x) &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_{\varepsilon+\sqrt{\varepsilon}}(x)} \check{\partial}^{\varepsilon+\sqrt{\varepsilon}} f_t(x) + N_{L \cap \text{dom } f}(x) \right\} \\ &= \bigcap_{\substack{\eta > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\eta(x)} \check{\partial}^\eta f_t(x) + N_{L \cap \text{dom } f}(x) \right\}. \end{aligned}$$

■

Remark 1 Observe that formula (16) remains valid if instead of $\check{\partial}^\varepsilon f_t(x)$ we use the larger set

$$\widehat{\partial}^\varepsilon f_t(x) := \left\{ y^* \in X^* \mid \begin{array}{l} \exists y \in X, \text{ such that } |f_t(y) - f_t(x)| \leq \varepsilon, \\ y^* \in \partial f_t(y) \text{ and } |\langle y^*, y - x \rangle| \leq \varepsilon \end{array} \right\}. \quad (20)$$

The reason is that this set is also included in $\partial_{3\varepsilon} f(x)$ (see the proof of (17)).

If X is finite-dimensional, $N_{\text{dom } f}(x) \subset N_{L \cap \text{dom } f}(x)$ for all $L \in \mathcal{F}(x)$ and (16) collapses to

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}^\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}. \quad (21)$$

In the following corollary we use an alternative enlargement of the Fenchel subdifferential, which involves both subgradients at nearby points y and ε -subgradients at the nominal point x . This provides a characterization of $\partial f(x)$ which is at the same time a refinement of formulas (6) and (16). For instance, the ε -subgradients of the f_t 's involved in (6) can be chosen in the range of ∂f_t at sufficiently f_t -graphically close points to x .

Corollary 7 *Under the assumptions of Theorem 6 we have, for every $x \in X$,*

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\},$$

where

$$\widehat{\partial}^\varepsilon f_t(x) := \left\{ y^* \in X^* \mid \begin{array}{l} \exists y \in B_\varepsilon(x) \text{ such that } |f_t(y) - f_t(x)| \leq \varepsilon \\ \text{and } y^* \in \partial f_t(y) \cap \partial_{2\varepsilon} f_t(x) \end{array} \right\}.$$

Proof. Let us first check that

$$\check{\partial}^\varepsilon f_t(x) \subset \widehat{\partial}^\varepsilon f_t(x).$$

If $x^* \in \check{\partial}^\varepsilon f_t(x)$, then $x^* \in \partial f_t(y)$ for some $y \in B_\varepsilon(x)$ such that $|f_t(y) - f_t(x)| \leq \varepsilon$ and $|\langle x^*, y - x \rangle| \leq \varepsilon$. So, for every $z \in \text{dom } f_t$,

$$\begin{aligned} \langle x^*, z - x \rangle &= \langle x^*, z - y \rangle + \langle x^*, y - x \rangle \\ &\leq f_t(z) - f_t(y) + \varepsilon \\ &\leq f_t(z) - f_t(x) + 2\varepsilon, \end{aligned}$$

and we deduce that $x^* \in \partial_{2\varepsilon} f_t(x)$; that is, $x^* \in \widehat{\partial}^\varepsilon f_t(x)$. Then, according to Theorem 6 we get the direct inclusion “ $\subset$ ”.

To check the converse inclusion “ $\supset$ ” we prove now that

$$\bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) \subset \partial_{3\varepsilon} f(x). \quad (22)$$

To this aim, take $x^* \in \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x)$. Then, there is some $t_0 \in T_\varepsilon(x)$ and $y \in B_\varepsilon(x)$ such that $x^* \in \partial f_{t_0}(y) \cap \partial_{2\varepsilon} f_{t_0}(x)$, with $|f_{t_0}(y) - f_{t_0}(x)| \leq \varepsilon$. Then for every $z \in \text{dom } f$

$$\begin{aligned} \langle x^*, z - x \rangle &\leq \langle x^*, z - y \rangle + \langle x^*, y - x \rangle \\ &\leq f_{t_0}(z) - f_{t_0}(y) + f_{t_0}(y) - f_{t_0}(x) + 2\varepsilon \\ &= f_{t_0}(z) - f_{t_0}(x) + 2\varepsilon \\ &\leq f(z) - f(x) + 3\varepsilon, \end{aligned}$$

that is, $x^* \in \partial_{3\varepsilon} f(x)$.

Thanks to (22) we write

$$\begin{aligned} \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}^\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\} &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \{ \partial_{3\varepsilon} f(x) + N_{L \cap \text{dom } f}(x) \} \\ &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \partial_{3\varepsilon} (f + I_L)(x) = \partial f(x). \end{aligned}$$

■

The following example illustrates the need for the lower semicontinuity assumption in Theorem 6, as well as for the condition $|f_t(y) - f_t(x)| \leq \varepsilon$.

Example 1 Let $X = \mathbb{R}$ and consider the functions $f_1, f_2 : X \rightarrow \mathbb{R}$ as

$$f_1(x) = \begin{cases} +\infty, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ x, & \text{if } x > 0, \end{cases} \quad \text{and } f_2(x) = \begin{cases} -x, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ +\infty, & \text{if } x > 0. \end{cases}$$

So, $f = \max\{f_1, f_2\} = 1 + I_{\{0\}}$ so that $\partial f(0) = \mathbb{R}$, but for small $\varepsilon > 0$

$$\check{\partial}^\varepsilon f_1(0) = \partial f_1(0) = \emptyset, \quad \check{\partial}^\varepsilon f_2(0) = \partial f_2(0) = \emptyset,$$

and Theorem 6 fails.

Example 2 Let X be any infinite-dimensional Banach space, let g be a non-continuous linear mapping, and define the functions $f_t : X \rightarrow \mathbb{R}$, $t \in T :=]0, +\infty[$, as

$$f_t(x) := tg(x).$$

So, $f := \sup_{t \in T} f_t = I_{[g \leq 0]}$ and we get $\partial f(\theta) = N_{[g \leq 0]}(\theta) \neq \emptyset$. But $\partial f_t \equiv t\partial g \equiv \emptyset$, so that $\check{\partial}^\varepsilon f_t(\theta) \equiv \emptyset$, for all $t \in T$, and Theorem 6 fails again.

The following example (e.g., [11]) shows that it is necessary to consider exact subdifferentials at nearby points.

Example 3 Let $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ be defined by

$$f_1(x) = \begin{cases} -\sqrt{x}, & \text{if } x \geq 0, \\ +\infty, & \text{if } x < 0, \end{cases} \quad \text{and } f_2(x) = f_1(-x).$$

Then, $f := \max\{f_1, f_2\} = I_{\{0\}}$ so that $\partial f(0) = \mathbb{R}$. But, $\partial f_1(0) = \partial f_2(0) = \emptyset$ and so, for every $\varepsilon > 0$, we have $T_\varepsilon(\theta) = \{1, 2\}$ and

$$\bigcap_{\varepsilon > 0} \overline{\text{co}} \{(\partial f_1(0) \cup \partial f_2(0)) + N_{\text{dom } f}(0)\} = \emptyset.$$

So, in order to recover the whole set $\partial f(0)$, we need to consider the subdifferentials of f_1 and f_2 at nearby points. Indeed, for $\varepsilon < 1$ we have

$$\begin{aligned} \check{\partial}^\varepsilon f_1(0) &= \left\{ y^* \in X^* \mid \begin{array}{l} \exists |y| \leq \varepsilon \text{ such that } |f_1(y)| \leq \varepsilon, \\ y^* \in \partial f_1(y) \text{ and } |y^*, y| \leq \varepsilon \end{array} \right\} \\ &= \left\{ -\frac{1}{2\sqrt{y}} \mid 0 < y \leq \varepsilon^2 \right\} = \left] -\infty, -\frac{1}{2\varepsilon} \right], \end{aligned}$$

and, similarly,

$$\check{\partial}^\varepsilon f_2(0) = \left[\frac{1}{2\varepsilon}, +\infty \right[.$$

Since $T_\varepsilon(0) = \{1, 2\}$,

$$\bigcup_{t \in T_\varepsilon(0)} \check{\partial}^\varepsilon f_t(0) + N_{\text{dom } f}(0) = \left] -\infty, -\frac{1}{2\varepsilon} \right] \cup \left[\frac{1}{2\varepsilon}, +\infty \right[+ \mathbb{R} = \mathbb{R}.$$

We close this section with the following example showing that, in general, Theorem 6 is not valid outside the Banach spaces setting.

Example 4 Let X be a lcs space such that there exists a proper lsc convex function $g \in \Gamma_0(X)$ having an empty subdifferential everywhere. This is the case of some Fréchet spaces ([16]), and also of some non-complete normed spaces (actually certain subspaces of $l^2(\mathbb{N})$ [1]).

According to [16], we may suppose that $\theta \in \text{dom } g$ and $g(\theta) = 0$. For $t \in T :=]0, +\infty[$, we define the function $f_t \in \Gamma_0(X)$ as

$$f_t(x) := tg(x),$$

so that $f := \sup_{t \in T} f_t = I_{[g \leq 0]}$. Since $\partial f_t \equiv t\partial f \equiv \emptyset$, the right-hand set in (16) is empty, whereas

$$\partial f(\theta) = N_{[g \leq 0]}(\theta) \neq \emptyset.$$

4 The case of locally convex spaces

We give in this section a general free-continuity formula characterizing the subdifferential of the supremum function in the setting of a lcs X . In what follows, $\mathcal{P}$ is the family of continuous seminorms in X .

We adapt the mapping introduced in (15) to our current setting: given a convex function $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $x \in \text{dom } \varphi$, for $p \in \mathcal{P}$ we denote

$$\check{\partial}_p^\varepsilon \varphi(x) := \left\{ y^* \in X^* \mid \begin{array}{l} \exists y \in X \text{ with } p(y - x) \leq \varepsilon, \text{ such that} \\ |\varphi(y) - \varphi(x)| \leq \varepsilon, y^* \in \partial\varphi(y) \text{ and } |\langle y^*, y - x \rangle| \leq \varepsilon \end{array} \right\}. \quad (23)$$

Also now, when $x \notin \text{dom } \varphi$ we set $\check{\partial}_p^\varepsilon \varphi(x) = \emptyset$.

Remark 2 It is worth observing that $\check{\partial}_p^\varepsilon \varphi(x)$ remains the same if we replace the inequality $|\langle y^*, y - x \rangle| \leq \varepsilon$ by $\langle y^*, y - x \rangle \leq \varepsilon$, thanks to the fact that $y^* \in \partial\varphi(y)$ and $|\varphi(y) - \varphi(x)| \leq \varepsilon$.

We give the main theorem of this section.

Theorem 8 *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t \in T} f_t$. Then for every $x \in X$*

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(x) \right\}. \quad (24)$$

Proof. If $x \notin \text{dom } f$, then $x \notin \text{dom } f_t$ for $t \in T_\varepsilon(x)$; hence, $x \notin \text{dom}(f_t + \text{I}_{\overline{L \cap \text{dom } f}})$ and the sets in both sides of (24) are empty.

Now we assume that $x \in \text{dom } f$. First we verify the inclusion “ $\supset$ ” in (24). We take $L \in \mathcal{F}(x)$ and $y^* \in \check{\partial}_p^\varepsilon (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(x)$ for some $t \in T_\varepsilon(x)$. Let $y \in \overline{L \cap \text{dom } f}$ be such that $p(y - x) \leq \varepsilon$,

$$\left| (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(y) - (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(x) \right| = |f_t(y) - f_t(x)| \leq \varepsilon,$$

$y^* \in \partial(f_t + \text{I}_{\overline{L \cap \text{dom } f}})(y)$ and $|\langle y^*, y - x \rangle| \leq \varepsilon$. Then, for all $z \in L \cap \text{dom } f$,

$$\langle y^*, z - x \rangle = \langle y^*, z - y \rangle + \langle y^*, y - x \rangle \leq f_t(z) - f_t(y) + \varepsilon \leq f(z) - f(y) + 3\varepsilon,$$

and we get $y^* \in \partial_{3\varepsilon}(f + \text{I}_L)(x)$; that is, $\check{\partial}_p^\varepsilon (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(x) \subset \partial_{3\varepsilon}(f + \text{I}_L)(x)$. Thus, recalling Lemma 4,

$$\begin{aligned} \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon (f_t + \text{I}_{\overline{L \cap \text{dom } f}})(x) \right\} &\subset \bigcap_{L \in \mathcal{F}(x)} \bigcap_{\varepsilon > 0} \partial_{3\varepsilon}(f + \text{I}_L)(x) \\ &= \bigcap_{L \in \mathcal{F}(x)} \partial(f + \text{I}_L)(x) = \partial f(x). \end{aligned}$$

To prove the inclusion “ $\subset$ ” in (24) we shall proceed by steps.

Step 1. Observe that, due to the following relations

$$\partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) = \partial_\varepsilon f_t(x) + N_{\overline{L \cap \text{dom } f}}(x) \subset \partial_\varepsilon (f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x),$$

formula (6) implies that

$$\partial f(x) \subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon (f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \right\}. \quad (25)$$

Step 2. Now, we fix $p \in \mathcal{P}$, $U \in \mathcal{N}_{X^*}$, $L \in \mathcal{F}(x)$, and $\varepsilon > 0$. Let $\|\cdot\|$ be a norm in L such that $p \leq \|\cdot\|$ in such a subspace, and take into account that $L^* \simeq X^*/L^\perp$ so that $\mathcal{N}_{X^*} + L^\perp$ is a basis of θ -neighborhoods in $X^*/L^\perp$ (see, e.g., [7]). Choose $\eta > 0$ such that

$$\eta + \sqrt{\eta} \leq \varepsilon \text{ and } \sqrt{\eta} B_{L^*} \subset U + L^\perp, \quad (26)$$

where B_{L^*} is the unit ball in L^* .

For the sake of notational brevity, we shall denote

$$g_t := f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}}, \quad t \in T,$$

where L is our fixed subspace.

Take $x^* \in \partial_\eta g_t(x)$, with $t \in T_\eta(x)$. Then $x_{|L}^* \in \partial_\eta g_{t|L}(x)$, and by Proposition 5 we find $x_\eta \in L (\subset X)$ and $v_\eta^* \in \partial g_{t|L}(x_\eta)$ (hence, $x_\eta \in \overline{L \cap \text{dom } f}$), together with $\lambda_\eta \in [-1, 1]$ and $y^* \in B_{L^*}$ (hence, $\sqrt{\eta} y^* \in \sqrt{\eta} B_{L^*} \subset U + L^\perp$), such that

$$p(x_\eta - x) \leq \|x_\eta - x\| \leq \sqrt{\eta} \leq \varepsilon, \quad (27)$$

$$v_\eta^* := x_{|L}^* + \sqrt{\eta}(y^* + \lambda_\eta x_{|L}^*) = (1 + \lambda_\eta \sqrt{\eta}) x_{|L}^* + \sqrt{\eta} y^*,$$

$$|\langle v_\eta^*, x_\eta - x \rangle| \leq \eta + \sqrt{\eta} \leq \varepsilon,$$

and

$$|f_t(x_\eta) - f_t(x)| = |g_t(x_\eta) - g_t(x)| \leq \eta + \sqrt{\eta} \leq \varepsilon. \quad (28)$$

By the Hahn-Banach theorem we extend v_η^* to an $x_\eta^* \in X^*$; hence, in particular, we have $x_\eta^* \in \partial g_t(x_\eta)$ and $|\langle x_\eta^*, x_\eta - x \rangle| = |\langle v_\eta^*, x_\eta - x \rangle| \leq \varepsilon$, which together with (27) and (28) ensures that

$$x_\eta^* \in \check{\partial}_p^\varepsilon g_t(x). \quad (29)$$

Hence we have proved that

$$\partial_\eta g_t(x) \subset \check{\partial}_p^\varepsilon g_t(x), \text{ for all } t \in T_\eta(x).$$

Step 3. Moreover, for every $u \in L$ we obtain

$$\begin{aligned}
\langle (1 + \lambda_\eta \sqrt{\eta})x^*, u - x \rangle &= \left\langle (1 + \lambda_\eta \sqrt{\eta})x_{|L}^*, u - x \right\rangle \\
&= \left\langle (1 + \lambda_\eta \sqrt{\eta})x_{|L}^* - v_\eta^*, u - x \right\rangle + \langle v_\eta^*, u - x \rangle \\
&= \langle -\sqrt{\eta}y^*, u - x \rangle + \langle x_\eta^*, u - x \rangle \\
&= \langle x_\eta^* - \sqrt{\eta}y^*, u - x \rangle
\end{aligned}$$

that is, taking into account that $\sqrt{\eta}y^* \in \sqrt{\eta}B_{L^*} \subset U + L^\perp$ (by (26)), and using (29),

$$\begin{aligned}
(1 + \lambda_\eta \sqrt{\eta}) \langle x^*, u - x \rangle &= \langle x_\eta^* - \sqrt{\eta}y^*, u - x \rangle \\
&\leq \sigma_{\check{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x).
\end{aligned} \tag{30}$$

Observe that (30) can be extended to $u \in X$, i.e. we also have

$$(1 + \lambda_\eta \sqrt{\eta}) \langle x^*, u - x \rangle \leq \sigma_{\check{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x) \text{ for all } u \in X, \tag{31}$$

due to the fact that, for any $u \in X$,

$$\sigma_{\check{\partial}_p^\varepsilon g_t(x) + U + L^\perp}(u - x) = \sup\{\langle u^* + \ell^*, u - x \rangle \mid u^* \in \check{\partial}_p^\varepsilon g_t(x) + U, \ell^* \in L^\perp\},$$

and the expression on the right-hand is $+\infty$ when $u - x \notin L$, equivalently $u \notin L$.

Step 4. Now observe that

$$\check{\partial}_p^\varepsilon g_t(x) + L^\perp \subset \check{\partial}_p^\varepsilon g_t(x). \tag{32}$$

Indeed, if $z^* \in \check{\partial}_p^\varepsilon g_t(x)$ and $u^* \in L^\perp$, then by (23) there exists $z \in X$ such that $p(z - x) \leq \varepsilon$, $|g_t(z) - g_t(x)| \leq \varepsilon$, $z^* \in \partial g_t(z)$ and $|\langle z^*, z - x \rangle| \leq \varepsilon$; in particular, $z \in \text{dom } g_t \subset L$. Then,

$$\begin{aligned}
z^* + u^* &\in \partial g_t(z) + L^\perp \subset \partial(g_t + \mathbf{I}_L)(z) \\
&= \partial(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}} + \mathbf{I}_L)(z) = \partial g_t(z).
\end{aligned}$$

As $|\langle z^* + u^*, z - x \rangle| = |\langle z^*, z - x \rangle| \leq \varepsilon$ we deduce that $z^* + u^* \in \check{\partial}_p^\varepsilon g_t(x)$, yielding (32).

Step 5. Consequently, using (31) and (32), for all $z \in X$ (as $t \in T_\eta(x) \subset T_\varepsilon(x)$)

$$\begin{aligned}
(1 + \lambda_\eta \sqrt{\eta}) \langle x^*, z - x \rangle &\leq \sigma_{\check{\partial}_p^\varepsilon g_t(x) + U}(z - x) \\
&\leq \sigma_{\bigcup_{t \in T_\eta(x)} \check{\partial}_p^\varepsilon g_t(x) + U}(z - x) \\
&\leq \sigma_{\bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U}(z - x) \\
&= \sigma_{\overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\}}(z - x);
\end{aligned}$$

therefore

$$(1 + \lambda_\eta \sqrt{\eta}) x^* \in \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\}$$

and so, recalling that $x^* \in \partial_\eta g_t(x)$, with $t \in T_\eta(x)$,

$$\bigcup_{t \in T_\eta(x)} \partial_\eta g_t(x) \subset \Lambda_\eta \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\}, \quad (33)$$

where $\Lambda_\eta = \left[\frac{1}{1+\sqrt{\eta}}, \frac{1}{1-\sqrt{\eta}} \right]$.

Step 6. Now, from (33) and Lemma 1 which ensures that the set in the right-hand side above is closed and convex,

$$\begin{aligned}
\bigcap_{\substack{\delta > 0 \\ M \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\delta(x)} \partial_\delta (f_t + \mathbf{I}_{\overline{M \cap \text{dom } f}})(x) \right\} &\subset \overline{\text{co}} \left\{ \bigcup_{t \in T_\eta(x)} \check{\partial}_p^\eta g_t(x) \right\} \\
&\subset \Lambda_\eta \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\},
\end{aligned}$$

so that, intersecting over $\varepsilon > 0$ (recall (25) and Lemma (2)),

$$\partial f(x) \subset \bigcap_{\varepsilon > 0} \Lambda_\eta \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\} = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) + U \right\},$$

which in turn gives us, by intersecting over U (recall (14)),

$$\partial f(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon g_t(x) \right\}.$$

Finally, the inclusion follows since L and p were arbitrarily chosen. $\blacksquare$

Remark 3 (24) gives an alternative formula of (16) in the setting of lcs spaces. If the underlying space is Banach, all that can be derived from (24), by using Proposition 5, is

$$\check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \subset \check{\partial}_p^\varepsilon f_t(y_1) + \mathbf{N}_{\overline{L \cap \text{dom } f}}(y_2),$$

for some y_1 and y_2 close to x . In this sense, we can not directly deduce (16) from (24) (in Banach spaces).

Remark 4 Similar to the observation made in Remark 1, we can equivalently write formula (24) as

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \right\},$$

where $\widehat{\partial}^\varepsilon$ is the mapping introduced in (20).

Remark 5 As in Corollary 7, we can show that

$$\check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \subset \widehat{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x),$$

where for a function $\varphi \in \Lambda(X)$ we denote

$$\widehat{\partial}_p^\varepsilon \varphi(x) := \left\{ y^* \in X^* \mid \begin{array}{l} \exists y \in X \text{ such that } p(y - x) \leq \varepsilon, |\varphi(y) - \varphi(x)| \leq \varepsilon \\ \text{and } y^* \in \partial \varphi(y) \cap \partial_{2\varepsilon} \varphi(x) \end{array} \right\}. \quad (34)$$

As a consequence of (34) we have that

$$\check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \subset \widehat{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \subset \partial_{2\varepsilon}(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x). \quad (35)$$

Then similar arguments to those used in Corollary 7 give rise to

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \widehat{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \right\}.$$

Once again the intersection over the L 's in (24) can be removed in the finite-dimensional setting. In the following theorem we show that this is also the case in a more general setting.

Theorem 9 *Under the assumptions of Theorem 8, if $\text{ri}(\text{dom } f) \neq \emptyset$ and $f|_{\text{aff}(\text{dom } f)}$ is continuous on $\text{ri}(\text{dom } f)$, then for all $x \in X$*

$$\partial f(x) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) \right\}.$$

Proof. Fix $x \in \text{dom } f$ and choose $L \in \mathcal{F}(x)$ such that $L \cap \text{ri}(\text{dom } f) \neq \emptyset$; hence by the accessibility lemma

$$\overline{L \cap \text{dom } f} = L \cap \overline{\text{dom } f}. \quad (36)$$

Next, given $\varepsilon > 0$, $p \in \mathcal{P}$ and $t \in T_\varepsilon(x)$, we show that

$$\check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \subset \text{cl} \left(\check{\partial}_p^{2\varepsilon}(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) + L^\perp \right). \quad (37)$$

Take $y^* \in \check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x)$ and let $y \in \overline{L \cap \text{dom } f}$ ($\subset L$) be such that $p(y - x) \leq \varepsilon$, $|f_t(y) - f_t(x)| \leq \varepsilon$, $|\langle y^*, y - x \rangle| \leq \varepsilon$, and $y^* \in \partial(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(y)$.

Denote $h_t := f_t + \mathbf{I}_{\overline{\text{dom } f}}$, $t \in T$. Since $\text{dom } f \subset \text{dom } f_t \cap \overline{\text{dom } f} = \text{dom } h_t \subset \overline{\text{dom } f}$, we have that

$$\text{ri}(\text{dom } f) = \text{ri}(\text{dom } f_t \cap \overline{\text{dom } f}) = \text{ri}(\text{dom } h_t) \text{ and } \text{aff}(\text{dom } h_t) = \text{aff}(\text{dom } f).$$

Consequently, because $h_t|_{\text{aff}(\text{dom } g_t)} \leq f|_{\text{aff}(\text{dom } f)}$ the function $h_t|_{\text{aff}(\text{dom } g_t)}$ is continuous on $\text{ri}(\text{dom } h_t)$ and so, according to [5, Theorem 15] (remember (36)),

$$y^* \in \partial(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(y) = \partial(h_t + \mathbf{I}_L)(y) = \text{cl}(\partial h_t(y) + L^\perp).$$

Let $U_0 \in \mathcal{N}_{X^*}$ be such that $\sigma_{U_0}(y - x) \leq \varepsilon$. Then for every $U \in \mathcal{N}_{X^*}$ such that $U \subset U_0$ there exists $z^* \in \partial h_t(y)$ such that $y^* \in z^* + L^\perp + U$ and so, as $x, y \in L$,

$$|\langle z^*, y - x \rangle| \leq |\langle y^*, y - x \rangle| + \sigma_U(y - x) \leq |\langle y^*, y - x \rangle| + \sigma_{U_0}(y - x) \leq 2\varepsilon,$$

showing that $z^* \in \check{\partial}_p^{2\varepsilon} h_t(x)$ and so,

$$y^* \in z^* + L^\perp + U \subset \check{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U.$$

This leads us to

$$\begin{aligned} \check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) &\subset \bigcap_{U \in \mathcal{N}_{X^*}, U \subset U_0} \left(\check{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U \right) \\ &= \bigcap_{U \in \mathcal{N}_{X^*}} \left(\check{\partial}_p^{2\varepsilon} h_t(x) + L^\perp + U \right) \\ &= \text{cl} \left(\check{\partial}_p^{2\varepsilon} h_t(x) + L^\perp \right), \end{aligned}$$

and (37) follows.

Now, by applying (24), and using (37), we obtain

$$\begin{aligned}
\partial f(x) &= \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon(f_t + \mathbf{I}_{\overline{L \cap \text{dom } f}})(x) \right\} \\
&\subset \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \text{cl} \left(\check{\partial}_p^{2\varepsilon}(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) + L^\perp \right) \right) \\
&\subset \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left(\text{cl} \left(\bigcup_{t \in T_{2\varepsilon}(x)} \check{\partial}_p^{2\varepsilon}(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) + L^\perp \right) \right) \\
&= \bigcap_{\substack{\varepsilon > 0, p \in \mathcal{P} \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left(\bigcup_{t \in T_{2\varepsilon}(x)} \check{\partial}_p^{2\varepsilon}(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) + L^\perp \right) \\
&= \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left(\bigcup_{t \in T_{2\varepsilon}(x)} \check{\partial}_p^{2\varepsilon}(f_t + \mathbf{I}_{\overline{\text{dom } f}})(x) \right),
\end{aligned}$$

where in the last equality we used Lemma (14). ■

5 The continuous case

In this section we get reduced versions of the previous formulas, (16) and (24), when continuity assumptions are imposed. We work in the general setting of a lcs space X , where $\mathcal{P}$ is the family of continuous seminorms in X . The main result comes next.

Theorem 10 *Let $f_t \in \Lambda(X)$, $t \in T$, and $f = \sup_{t \in T} f_t$. Assume that f is finite and continuous at some point. Then for every $x \in X$*

$$\partial f(x) = N_{\text{dom } f}(x) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(x)} \check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \right\}, \quad (38)$$

where

$$\overline{T}_\varepsilon(x) := \{t \in T \mid (\text{cl } f_t)(x) \geq f(x) - \varepsilon\}.$$

Proof. We assume without loss of generality that $x = \theta \in \text{dom } f$ and $f(\theta) = 0$. We fix $x^* \in \partial f(\theta)$ and let $x_0 \in \text{dom } f$ be such that f is continuous at x_0 , that is, there exist $W \in \mathcal{N}_X$ and $m \in \mathbb{R}_+$ such that

$$x_0 + W \subset \text{dom } f \quad \text{and} \quad \sup_{t \in T, w \in W} f_t(x_0 + w) \leq m; \quad (39)$$

hence, all the f_t 's are continuous at x_0 [19, Lemma 2.2.8].

Step 1. Let us first suppose that all the f_t 's are lsc, so that $\overline{T}_\varepsilon(x) = T_\varepsilon(x)$. Fix $\varepsilon \in]0, 1[$ and $p \in \mathcal{P}$.

We introduce the continuous seminorm $\tilde{p} := \max\{p, p_W\}$, where p_W is the Minkowski gauge of W . Then, by Theorem 9,

$$\partial f(\theta) = \bigcap_{\delta > 0, q \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\delta(\theta)} \check{\partial}_q^\delta(f_t + \text{I}_{\overline{\text{dom } f}})(\theta) \right\} \subset \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_{\tilde{p}}^\varepsilon(f_t + \text{I}_{\overline{\text{dom } f}})(\theta) \right\}. \quad (40)$$

Step 2. Choose $V_0 \in \mathcal{N}_{X^*}$ satisfying

$$\sigma_{V_0}(x_0) \leq 1, \quad (41)$$

and take $V \in \mathcal{N}_{X^*}$ such that $V \subset V_0$; hence, $\sigma_V(x_0) \leq \sigma_{V_0}(x_0) \leq 1$, and by (40)

$$x^* \in \text{co} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_{\tilde{p}}^\varepsilon(f_t + \text{I}_{\overline{\text{dom } f}})(\theta) \right\} + V.$$

In other words, there exist $k \in \mathbb{N}$, $(\lambda_1, \dots, \lambda_k) \in \Delta_k$, $t_1, \dots, t_k \in T_\varepsilon(\theta)$, and $z_i^* \in \check{\partial}_{\tilde{p}}^\varepsilon(f_{t_i} + \text{I}_{\overline{\text{dom } f}})(\theta)$, $i \in \overline{1, k}$, such that

$$x^* \in \lambda_1 z_1^* + \dots + \lambda_k z_k^* + V.$$

Moreover, taking into account Moreau-Rockafellar sum rule (recall that the f_t 's are continuous at $x_0 \in \text{dom } f$), and using the definition of $\check{\partial}_{\tilde{p}}^\varepsilon$, for each $i \in \overline{1, k}$ there exist $y_i \in \overline{\text{dom } f}$, with $\tilde{p}(y_i) \leq \varepsilon$, and elements $y_i^* \in \partial f_{t_i}(y_i)$ and $v_i^* \in N_{\overline{\text{dom } f}}(y_i)$ such that $z_i^* = y_i^* + v_i^*$,

$$|\langle y_i^* + v_i^*, y_i \rangle| \leq \varepsilon, \quad (42)$$

and

$$|f_{t_i}(y_i) - f_{t_i}(\theta)| = |(f_{t_i} + \text{I}_{\overline{\text{dom } f}})(y_i) - (f_{t_i} + \text{I}_{\overline{\text{dom } f}})(\theta)| \leq \varepsilon; \quad (43)$$

consequently, y_i and y_i^* satisfy the following relations

$$y_i^* \in \partial_{3\varepsilon} f(\theta) \text{ and } |\langle y_i^*, y_i \rangle| \leq \varepsilon, \quad (44)$$

and

$$y_i^* \in \check{\partial}_{\tilde{p}}^\varepsilon f_{t_i}(\theta). \quad (45)$$

Indeed, for any $u \in \text{dom } f$ the information we have above on y_i and y_i^* leads, on one hand, to

$$\begin{aligned}\langle y_i^*, u \rangle &= \langle y_i^*, u - y_i \rangle + \langle y_i^* + v_i^*, y_i \rangle + \langle v_i^*, -y_i \rangle \\ &\leq f_{t_i}(u) - f_{t_i}(y_i) + \varepsilon \\ &\leq f(u) - f_{t_i}(\theta) + 2\varepsilon \\ &\leq f(u) - f(\theta) + 3\varepsilon,\end{aligned}$$

which shows that $y_i^* \in \partial_{3\varepsilon} f(\theta)$. The second inequality in (44) also follows since

$$\langle y_i^*, -y_i \rangle \leq f_{t_i}(\theta) - f_{t_i}(y_i) \leq \varepsilon,$$

and, using (42),

$$\langle y_i^*, y_i \rangle = \langle y_i^* + v_i^*, y_i \rangle + \langle v_i^*, -y_i \rangle \leq \varepsilon.$$

This shows that $|\langle y_i^*, y_i \rangle| \leq \varepsilon$, which together with (43), and the relations $\tilde{p}(y_i) \leq \varepsilon$ and $y_i^* \in \partial f_{t_i}(y_i)$, lead us to (45).

We also have

$$v_i^* \in N_{\text{dom } f}^{2\varepsilon}(\theta). \quad (46)$$

Actually, for any $u \in \text{dom } f$, by (42) and (44) we have

$$\begin{aligned}\langle v_i^*, u \rangle &= \langle v_i^*, u - y_i \rangle + \langle v_i^*, y_i \rangle \leq \langle v_i^*, y_i \rangle \\ &= \langle v_i^* + y_i^*, y_i \rangle - \langle y_i^*, y_i \rangle \leq 2\varepsilon,\end{aligned}$$

and this yields (46).

Also, since $p_W(y_i) \leq \tilde{p}(y_i) \leq \varepsilon$, we infer that $y_i \in \varepsilon W$ and so, using (39),

$$x_0 + y_i \in x_0 + \varepsilon W \subset x_0 + W \subset \text{dom } f,$$

that is, since $v_i^* \in N_{\text{dom } f}(y_i)$,

$$\langle v_i^*, x_0 \rangle = \langle v_i^*, (x_0 + y_i) - y_i \rangle \leq 0. \quad (47)$$

Step 3. Now, we denote

$$y_V^* := \lambda_1 y_1^* + \cdots + \lambda_k y_k^*, \quad v_V^* := \lambda_1 v_1^* + \cdots + \lambda_k v_k^*;$$

hence, by (44), (47) and (45),

$$y_V^* \in \partial_{3\varepsilon} f(\theta) \cap \text{co} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_p^\varepsilon f_t(\theta) \right\}, \quad (48)$$

and

$$v_V^* \in N_{\text{dom } f}^{2\varepsilon}(\theta), \quad \langle v_V^*, x_0 \rangle \leq 0, \quad (49)$$

at the same time as

$$v_V^* \in x^* - y_V^* + V, \quad (50)$$

entailing that, due to (41),

$$\begin{aligned} \langle y_V^*, x_0 \rangle &= \langle y_V^* + v_V^*, x_0 \rangle - \langle v_V^*, x_0 \rangle \\ &\geq \langle x^*, x_0 \rangle - \sigma_V(x_0) \\ &\geq \langle x^*, x_0 \rangle - \sigma_{V_0}(x_0) \geq \langle x^*, x_0 \rangle - 1. \end{aligned} \quad (51)$$

Consequently, since $y_V^* \in \partial_{3\varepsilon} f(\theta)$, we obtain that, for all $w \in W$,

$$\langle y_V^*, x_0 + w \rangle \leq f(x_0 + w) + 3\varepsilon \leq m + 3\varepsilon, \quad (52)$$

and so, because of (51), we find some $r > 0$ such that

$$\langle y_V^*, w \rangle \leq -\langle y_V^*, x_0 \rangle + m + 3\varepsilon \leq -\langle x^*, x_0 \rangle + 1 + m + 3\varepsilon \leq r; \quad (53)$$

that is, considering the natural partial order in $\mathcal{N}_{X^*}$, by Alaoglu-Bourbaki theorem the net $(y_V^*)_{V \in \mathcal{N}_{X^*}, V \subset V_0}$ is contained in the weak*-compact set rW° . Hence, we may assume that $(y_V^*)_V$ weak*-converges to some element $y_\varepsilon^* \in X^*$ such that (recall (48))

$$y_\varepsilon^* \in rW^\circ \cap \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_p^\varepsilon f_t(\theta) \right\} \subset rW^\circ \cap \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon f_t(\theta) \right\}. \quad (54)$$

It also follows that the corresponding (sub)net $(v_V^*)_{V \in \mathcal{N}_{X^*}, V \subset V_0}$ weak*-converges to the element $v_\varepsilon^* := x^* - y_\varepsilon^*$ and that $v_\varepsilon^* \in N_{\frac{2\varepsilon}{\text{dom } f}}(\theta)$. Indeed, given $U_1 \in \mathcal{N}_{X^*}$, we choose $U_0 \in \mathcal{N}_{X^*}$ such that $U_0 + U_0 \subset U_1$. Since $U := v_\varepsilon^* + U_0 \in \mathcal{N}_{v_\varepsilon^*}$, we have $x^* - U \in \mathcal{N}_{y_\varepsilon^*}$ and so, there is some $V_1 \in \mathcal{N}_{X^*}$ such that $y_{V'}^* \in x^* - U$ for all $V' \subset V_1$; that is, $\theta \in y_{V'}^* - x^* + U$ and, using (50),

$$v_{V'}^* \in v_{V'}^* + y_{V'}^* - x^* + U \subset V' + U.$$

Then we get, since $V' \cap U_0 \subset V'$ and $U = v_\varepsilon^* + U_0$,

$$v_{V'}^* \in V' \cap U_0 + U = v_\varepsilon^* + V' \cap U_0 + U_0 \subset v_\varepsilon^* + U_1,$$

showing that $(v_V^*)_{V \in \mathcal{N}_{X^*}, V \subset V_0}$ weak*-converges to $v_\varepsilon^* \in N_{\frac{2\varepsilon}{\text{dom } f}}(\theta)$, by (49).

Step 4. Again, since $(y_\varepsilon^*)_\varepsilon \subset rW^\circ$ we may assume that the net $(y_\varepsilon^*)_\varepsilon$ weak*-converge to some y^* , which by (54) satisfies

$$y^* \in \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_p^\varepsilon f_t(\theta) \right\}.$$

Moreover, since p was arbitrarily fixed in $\mathcal{P}$, we deduce that

$$y^* \in \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_p^\varepsilon f_t(\theta) \right\}.$$

In addition, since $v_\varepsilon^* = x^* - y_\varepsilon^*$, by (49) it follows that $(v_\varepsilon^*)_\varepsilon$ also weak*-converges to

$$v^* = x^* - y^* \in N_{\overline{\text{dom } f}}(\theta) = N_{\text{dom } f}(\theta).$$

Consequently,

$$x^* = v^* + y^* \in N_{\text{dom } f}(\theta) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_p^\varepsilon f_t(\theta) \right\},$$

yielding the inclusion “ $\subset$ ”.

Step 5. We consider the case when the f_t 's are not necessarily lsc. Due to the continuity assumption on f , entailing the continuity of the f_t 's at x_0 , according to [11, Corollary 9(ii)] we have

$$\text{cl } f = \sup_{t \in T} \text{cl } f_t.$$

If $x \in \text{dom } f$ is such that $\partial f(x) \neq \emptyset$, then $\partial f(x) = \partial(\text{cl } f)(x)$ and $f(x) = (\text{cl } f)(x)$. Thus, from Step 4 applied to the $(\text{cl } f_t)$'s, we obtain

$$\partial f(x) = \partial(\text{cl } f)(x) = N_{\text{dom}(\text{cl } f)}(x) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(x)} \check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \right\},$$

and the inclusion “ $\subset$ ” follows since $N_{\text{dom}(\text{cl } f)}(x) \subset N_{\text{dom } f}(x)$.

Step 6. We show the opposite inclusion “ $\supset$ ”. Given $x \in \text{dom } f (\subset \text{dom } f_t)$, by (35) we have $\check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \subset \partial_{2\varepsilon}(\text{cl } f_t)(x)$. Then, since it can be easily proved that $\partial_{2\varepsilon}(\text{cl } f_t)(x) \subset \partial_{3\varepsilon} f(x)$ for all $t \in \overline{T}_\varepsilon(x)$, we deduce that for all $\varepsilon > 0$ and $p \in \mathcal{P}$

$$\overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(x)} \check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \right\} \subset \partial_{3\varepsilon} f(x).$$

Therefore

$$\begin{aligned} N_{\text{dom } f}(x) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(x)} \check{\partial}_p^\varepsilon(\text{cl } f_t)(x) \right\} &\subset N_{\text{dom } f}(x) + \bigcap_{\varepsilon > 0} \partial_\varepsilon f(x) \\ &= N_{\text{dom } f}(x) + \partial f(x) = \partial f(x). \end{aligned}$$

■

Remark 6 As can be observed from the proof of Theorem 10, the continuity assumption on f used there can be weakened to the existence of small $\varepsilon > 0$ such that the function $\sup_{t \in \overline{T}_\varepsilon(x)} f_t$ is finite and continuous at some point.

The following corollary is straightforward from Theorem 10.

Corollary 11 *Let $f_t \in \Gamma_0(X)$, $t \in T$, and $f = \sup_{t \in T} f_t$. Assume that f is finite and continuous at some point. Then for every $x \in X$*

$$\partial f(x) = N_{\text{dom } f}(x) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \check{\partial}_p^\varepsilon f_t(x) \right\}.$$

We obtain the classical result of Valadier [18, Theorem 1]:

Corollary 12 *Let $f_t \in \Lambda(X)$, $t \in T$, and $f = \sup_{t \in T} f_t$. Assume that f is continuous at $x \in \text{dom } f$. Then*

$$\partial f(x) = \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x), p(y-x) \leq \varepsilon} \partial f_t(y) \right\}.$$

Proof. We assume again that $x = \theta \in \text{dom } f$ and $f(\theta) = 0$. Since f is continuous at θ , by Theorem 10 we have

$$\begin{aligned} \partial f(\theta) &= N_{\text{dom } f}(\theta) + \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon (\text{cl } f_t)(\theta) \right\} \\ &= \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in \overline{T}_\varepsilon(\theta)} \check{\partial}_p^\varepsilon (\text{cl } f_t)(\theta) \right\}. \end{aligned} \quad (55)$$

On one hand, the continuity of f at θ implies the continuity of all the f_t 's at θ . So,

$$(\text{cl } f_t)(\theta) = f_t(\theta) \text{ for all } t \in T,$$

and we get

$$\overline{T}_\varepsilon(\theta) = T_\varepsilon(\theta). \quad (56)$$

On the other hand, there exist $U \in \mathcal{N}_X$ and $m \geq 0$ such that, for all $u \in U$,

$$f_t(u) \leq f(u) \leq m, \quad (57)$$

and so, by [19, Lemma 2.2.8],

$$|f_t(u) - f_t(\theta)| \leq mp_U(u). \quad (58)$$

This entails the continuity of the f_t 's at the points of U . Hence, (55) together with (56) give us, for all $p \in \mathcal{P}$,

$$\partial f(\theta) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta)} \check{\partial}_{\tilde{p}}^\varepsilon(\text{cl } f_t)(\theta) \right\},$$

where $\tilde{p} := \max\{p, p_U\}$. Observe that if $y^* \in \check{\partial}_{\tilde{p}}^\varepsilon(\text{cl } f_t)(\theta)$, for $t \in T_\varepsilon(\theta)$, then there exists $y \in X$ satisfying $\tilde{p}(y) \leq \varepsilon$ such that $y^* \in \partial(\text{cl } f_t)(y)$. But $p_U(y) \leq \varepsilon$ ensures that $y \in \varepsilon U \subset U$, and so, by (57), f_t is continuous at y so that $y^* \in \partial(\text{cl } f_t)(y) = \partial f_t(y)$. This gives rise to

$$\partial f(\theta) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta), p(y) \leq \varepsilon} \partial f_t(y) \right\},$$

and the inclusion " $\subset$ " follows by intersecting over $p \in \mathcal{P}$.

To show the converse inclusion " $\supset$ ", we take

$$x^* \in \bigcap_{\varepsilon > 0, p \in \mathcal{P}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta), p(y) \leq \varepsilon} \partial f_t(y) \right\}.$$

Then, given $\varepsilon \in]0, \frac{1}{2}[$ and $p \in \mathcal{P}$, we get

$$x^* \in \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(\theta), \tilde{p}(y) \leq \varepsilon} \partial f_t(y) \right\}, \quad (59)$$

where $\tilde{p} := \max\{p, p_U\}$ ($\in \mathcal{P}$), with U being defined as in (57). Observe that if $y^* \in \partial f_t(y)$ for some $t \in T_\varepsilon(\theta)$ and $y \in X$ such that $\tilde{p}(y) \leq \varepsilon$, then $y \in \varepsilon U \subset \frac{1}{2}U$ and, taking into account (58),

$$|f_t(y) - f_t(\theta)| \leq m p_U(y) \leq m \tilde{p}(y) \leq m \varepsilon,$$

$$|f_t(2y) - f_t(\theta)| \leq m p_U(2y) \leq 2m \tilde{p}(y) \leq 2m \varepsilon.$$

Consequently, for every $z \in \text{dom } f$,

$$\begin{aligned} \langle y^*, z \rangle &= \langle y^*, z - y \rangle + \langle y^*, y \rangle \\ &= \langle y^*, z - y \rangle + \langle y^*, 2y - y \rangle \\ &\leq f_t(z) - f_t(y) + f_t(2y) - f_t(y) \\ &= (f_t(z) - f_t(\theta)) + (f_t(\theta) - f_t(y)) + (f_t(2y) - f_t(\theta)) + (f_t(\theta) - f_t(y)) \\ &\leq f_t(z) - f_t(\theta) + 4m \varepsilon, \end{aligned}$$

and we get, since $t \in T_\varepsilon(\theta)$,

$$\langle y^*, z \rangle \leq f(z) - f(\theta) + 4m \varepsilon + \varepsilon,$$

so that $y^* \in \partial_{(4m+1)\varepsilon} f(\theta)$. Thus, from (59) it follows that

$$x^* \in \bigcap_{0 < \varepsilon < \frac{1}{2}} \partial_{(4m+1)\varepsilon} f(\theta) = \partial f(\theta),$$

and the proof is complete. ■

References

- [1] J. BORWEIN AND D. W. TINGLEY, *On supportless convex sets*. Proc. Amer. Math. Soc. 94 (1985), no. 3, 471–476.
- [2] A. BRØNSTED, *On the subdifferential of the supremum of two convex functions*, Math. Scand., 31 (1972), pp. 225–230.
- [3] R. CORREA AND A. JOFRÉ, *Some properties of semismooth and regular functions in nonsmooth analysis*, Recent advances in system modelling and optimization (Santiago, 1984), 69–85, Lecture Notes in Control and Inform. Sci., 87, Springer, Berlin, 1986.
- [4] R. CORREA AND A. SEEGER, *Directional derivative of a minmax function*, Non-linear Anal. 9 (1985), no. 1, 14–22.
- [5] R. CORREA, A. HANTOUTE AND M. A. LÓPEZ, *Weaker conditions for subdifferential calculus of convex functions*. J. Funct. Anal. 271 (2016), 1177–1212.
- [6] J. M. DANSKIN, *The Theory of Max-Min and its Applications to Weapons Allocations Problems*, Springer, New York, 1967.
- [7] M. FABIAN, P. HABALA, P. HAJEK, V. MONTESINOS SANTALUCIA, J. PELANT AND V. ZIZLER, *Functional Analysis and Infinite-Dimensional Geometry*, Springer-Verlag New York, 2001.
- [8] J. FLORENCE AND M. LASSONDE, *Subdifferential estimate of the directional derivative, optimality criterion and separation principles*. Optimization 62 (2013), no. 9, 1267–1288.
- [9] A. HANTOUTE, *Subdifferential set of the supremum of lower semi-continuous convex functions and the conical hull intersection property*. Top 14 (2006), no. 2, 355–374.
- [10] A. HANTOUTE AND M. A. LÓPEZ, *A complete characterization of the subdifferential set of the supremum of an arbitrary family of convex functions*. J. Convex Anal. 15 (2008), no. 4, 831–858.
- [11] A. HANTOUTE, M. A. LÓPEZ AND C. ZĂLINESCU, *Subdifferential calculus rules in convex analysis: a unifying approach via pointwise supremum functions*. SIAM J. Optim. 19 (2008), no. 2, 863–882.

- [12] A. HANTOUTE AND A. SVENSSON, *General representation of δ -normal sets and normal cones to sublevel sets of convex functions*, submitted, 2016.
- [13] A. D. IOFFE, *A note on subdifferentials of pointwise suprema*. Top 20 (2012), 456-466.
- [14] C. LI AND K. F. NG, *Subdifferential Calculus Rules for Supremum Functions in Convex Analysis*. SIAM J. Optim., to appear.
- [15] O. LÓPEZ AND L. THIBAUT, *Sequential formula for subdifferential of upper envelope of convex functions*. J. Nonlinear Convex Anal. 14 (2013), no. 2, 377–388.
- [16] R. T. ROCKAFELLAR, *On the subdifferentiability of convex functions*. Proc. Amer. Math. Soc. 16 (1965), 605-611.
- [17] V. N. SOLOV'EV, *The subdifferential and the directional derivatives of the maximum of a family of convex functions*, Izv. Ross Akad. Nauk Ser. Mat., 65 (2001), pp. 107–132.
- [18] M. VALADIER, *Sous-différentiels d'une borne supérieure et d'une somme continue de fonctions convexes*, C. R. Acad. Sci. Paris Sér. A-B Math., 268 (1969), pp. 39–42.
- [19] C. ZĂLINESCU, *Convex analysis in general vector spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.